ISSN 2229-5518

The m-deficient number of complete bigraphs

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Abstract— An open problem posed by Aaron and Lewinter in [2] asks whether the m-deficient number interpolates or not. A negative answer of this problem is established in [5]. The counter example in [5] was obtained by characterizing the m-deficient number of complete graph. In this note another counter example was obtained by characterizing the m-deficient number of complete bipartite K_{n_1, n_2}, $n_1 \neq n_2$ in a similar way.

Index Terms— Key words: Interpolate, deficient number, Degree preserving
Subject classification: 05C99 (Graph Theory).

INTRODUCTION

An integer valued function of on the spanning tree set of a graph G is called interpolating if when ever an Integer k satisfies f(T)<k<f(T[1]) for spanning trees T and T[1], there exists a spanning tree T[11] such that f (T[11])=k. Thus interpolation may be thought of as a discrete analogue of the intermediate value property. A vertex v of Spanning tree T of a graph G is called degree preserving (DP) if $\deg^{v}_{T} = \deg^{v}_{G}$. It is shown in [4] that the number of degree preserving vertices interpolate over the spanning tree set of a graph. In [2], the concept of degree preserving is generalized as follows: A vertex v of spanning tree T of a graph G is called m-deficient if $\deg^{v}_{G} - \deg^{v}_{T} = m$. Note that a degree preserving vertex v is O-deficient. An integer k is an m-deficient number of graph G if there is a spanning tree T of G such that k=N (G, T, m). That is N (G, T, m) denote the k number of m-deficient vertices in T.

The set of deficiencies of spanning trees of a graph G may differ, it is shown in [2] that the sum of the deficiencies of the vertices of any spanning tree of G is invariant. If G has 'n' vertices and 'e' edges, the sum of the deficiencies in any spanning tree is 2(e-n+1). If G is a planar graph with f faces then by using Euler's planar graph formula the sum of deficiencies is 2(f-1).

An open problem posed by Aaron and Lewinter in [2] asks whether the m-deficient number interpolates or not. A negative answer of this problem is established in [5]. The counter example in [5] was obtained by characterizing the m-deficient number of complete graphs. In this note another counter example was obtained characterizing the m-deficient number of complete bipartite graphs.

RESULTS

The following Lemma is equivalent to 2.1.10 [3].

Lemma 1. Let $d_1, d_2, \ldots, d_{n_1}, d_{n_1+1}, \ldots, d_{n_1+n_2}$ be the degree sequence of a graph of order n_1+n_2. Then there is a spanning tree T of the complete bipartite graph K_{n_1,n_2} ($n_1 \neq n_2$) with vertex set $V=\{v_1,v_2,\ldots, v_{n_1+1},\ldots, v_{n_1+n_1}\}$ such that $\deg v^i = d_i$ for $1 \leq i \leq n_1+n_2$ if and only if

$$\sum_{i=1}^{n_1+n_2} d_i = 2(n_1+n_2-1)$$

The above Lemma 1 is useful to establish the following Theorem 1.

Theorem1.
Let m, n_1, n_2 be three integers such that $o \leq m \leq n_1+n_2-2$. Then an integer $k \geq 0$ is an m-deficient number of K_{n_1, n_2} if and only if

$$k \leq \frac{n_1+n_2-2}{n_i - m-1} \qquad k \neq n_1+n_2-3$$

Proof.

Obverse that K_{n_1, n_2} is a complete bigraph with degrees n1 and n_2. If k is an m-deficient number of K_{n_1, n_2} there is a spanning tree T of K_{n_1, n_2} such that there exist exactly k vertices in T with degree $n_i - m \geq 2$, $i=1$ or 2..
Hence we have $k(n_i - m) + (n_1+n_2-k) \leq 2(n_1+n_2) - 2$.
This means $k(n_i - m) - k \leq (n_1+n_2) - 2$
That is (1)

$$k \leq \frac{n_1+n_2-2}{n_i - m-1}$$

If $k = n_1 + n_2 - 3$ then by (1) and the inequality $n_i - 2 \geq m$, we deduce $m=n_i-2$ or $n_i-m=2$. By Lemma 1, there are positive integers $d_i \neq 2$ satisfying $1 \leq i \leq n_1+n_2-k=3$such that

$$2(n_1+n_2-3) + \sum_{i=1}^{3} d_i = 2(n_1+n_2-1)$$

Hence $\sum_{i=1}^{3} d_i = 4$ which is impossible since $d_i \neq 2$ for $1 \leq i \leq 3$

ISSN 2229-5518

Then consider following two cases

Case 1.
$m = n_i - 2$

In this case, $k \le n_1 + n_2 - 2$ and $k \ne n_1 + n_2 - 3$.The m-deficient vertices in a spanning tree T are precisely the vertices of degree 2 .When $k=n_1+n_2-2$, a Hamiltonian path of $K_{n1,\ n2}$ has k, m-deficient vertices. When $K \le n_1+n_2 - 4$,
we get $d_1=d_2=\ldots\ldots=d_k=2$, $d_{k+1}= n_1+n_2-k-1$ and
$d_{k+2} = d_{k+3}=\ldots..= d_{n_1+n_2}=1$
Obviously

$$\sum_{i=1}^{n_1+n_2} di = 2\ (n_1+n_2-1)$$

and, in light of Lemma 1, there is a spanning tree T of $K_{n_1,\ n_2}$ such that
$k=N\ (K_{n_1,\ n_2}, T\ ; m)$. Thus k is an m-deficient number of $K_{n_1,\ n_2}$

Case 2.
$m \le n_i -3$

In this case, $k \le (n_1+n_2-2)\ /\ (n_i - m-1) < n_1+n_2-3$
Let $j = k\ (n_i - m -2) +2$ Then $j \le n_1+n_2-k$ let
$d_1=\ldots\ldots=d_k=(n_i-m)\ge 2$, $d_{k+1}=\ldots\ldots= d_{k+j} =1$and $d_{k+j+1}=\ldots\ldots..= d_{n_1+n_2}=2$.

Then,

$$\sum_{i=1}^{n_1+n_2} d_i =k\ (n_i-m)+j+2(n_1+n_2-k-j)$$
$$=k\ (n_i-m-2) - j+2(n_1+n_2)$$
$$=2(n_1+n_2-1)$$

By Lemma 1, k is an m-deficient number of $K_{n_1,\ n_2}$
In the following corollary we given an another counter example is established in similar way in [1]

Corollary1.

[1] Let m and n be two integers such that
$o \le m \le n - 2$. Then an integer $k \ge o$ is an m-deficient number of $K_{n,n}$ if and only if

$$k \le \frac{2n-2}{n-m-1} \qquad \text{and } k \ne 2n - 3$$

Corollary 2.
If $o \le m \le n_i - 3$ then k is an m-deficient number of $K_{n_1,\ n_2}$ if and only if

$$k \le \frac{n_1+n_2-2}{n_i - m-1}$$

Corollary 3.

If $m = n_i -2$ then k is an m-deficient number of $K_{n_1,\ n_2}$ if and only if

$$o \le k \le n_1+n_2-2$$

Corollary 4.

For every positive integer m, the m-deficient number of $K_{m_1+2,\ m_1+2}$ does not interpolate.

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